PHYS 301 Tutorial #2 Group Problem Sol'ns.

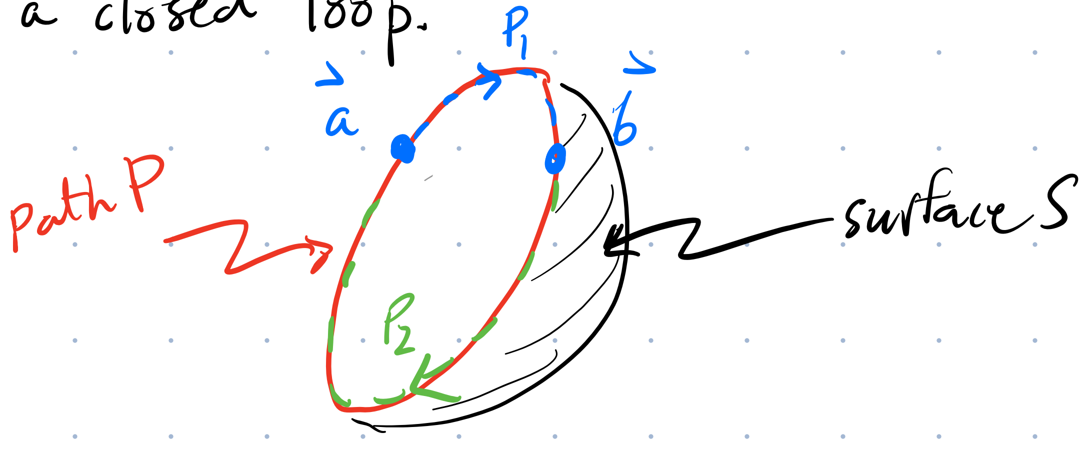
1. Recall Stoke's Theorem:

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \oint_P \vec{F} \cdot d\vec{l}$$

(a) Since given $\vec{\nabla} \times \vec{F} = 0$, must have:

$$\oint_P \vec{F} \cdot d\vec{l} = 0,$$

(b) Pick an arbitrary open surface bounded by a closed loop.



Pick two points $\vec{a}$ & $\vec{b}$ on boundary s.t.

$$I = \oint_P \vec{F} \cdot d\vec{l} = \int_{\vec{a}\,P_1}^{\vec{b}} \vec{F} \cdot d\vec{l} + \int_{\vec{b}\,P_2}^{\vec{a}} \vec{F} \cdot d\vec{l}$$

since $\vec{\nabla} \times \vec{F} = 0$, $I = 0$

$$\therefore \int_{\vec{a}\,P_1}^{\vec{b}} \vec{F} \cdot d\vec{l} = -\int_{\vec{b}\,P_2}^{\vec{a}} \vec{F} \cdot d\vec{l}$$

(Flip limits & reverse sign of right side integral)

$$\boxed{\therefore \int_{\vec{a}\,P_1}^{\vec{b}} \vec{F} \cdot d\vec{l} = \int_{\vec{a}\,P_2}^{\vec{b}} \vec{F} \cdot d\vec{l}}$$

(c) Since $\vec{\nabla} \times (\vec{\nabla} V) = 0 \ \forall \ V$ & given $\vec{\nabla} \times \vec{F} = 0$, must be possible to write

$$\vec{F} = \vec{\nabla} V \quad \text{or} \quad \vec{F} = -\vec{\nabla} V$$

In E&M we will find that $\vec{\nabla} \times \vec{E} = 0$ s.t. we can write $\vec{E} = -\vec{\nabla} V$ where V is the electric potential.

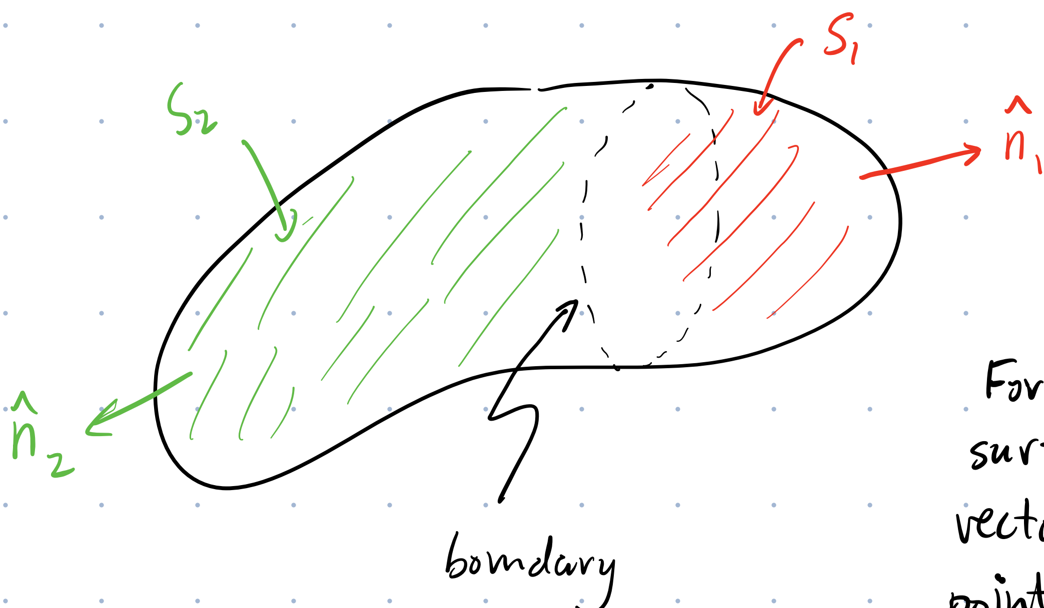
2. Recall the divergence theorem.

$$\int_V \vec{\nabla}\cdot\vec{F}\, d\tau = \oint_S \vec{F}\cdot d\vec{a}$$

(a) $\therefore$ if $\vec{\nabla}\cdot\vec{F} = 0$

$$\oint \vec{F}\cdot d\vec{a} = 0.$$

(b) Select an arbitrary volume / closed surface:



For a closed surface, unit vectors $\hat{n}$ point outwards.

Imagine dividing closed surface into two pieces that share a common boundary.

Then we can write:

$$I = \oint_S \vec{F} \cdot d\vec{a} = \int_{S_1} (\vec{F} \cdot \hat{n}_1) \cdot d\vec{a} + \int_{S_2} (\vec{F} \cdot \hat{n}_2) \cdot d\vec{a}$$

By the divergence theorem, $I = 0$ since $\vec{\nabla} \cdot \vec{F} = 0$

$$\therefore \int_{S_1} (\vec{F} \cdot \hat{n}_1) \cdot d\vec{a} = -\int_{S_2} (\vec{F} \cdot \hat{n}_2) \cdot d\vec{a}$$

$$= \int_{S_2} (\vec{F} \cdot (-\hat{n}_2)) \cdot d\vec{a}$$

For surfaces S_1 & S_2, unit vectors should point in same dir'ns so that $\vec{F} \cdot \hat{n}$ has the same sign.

$$\therefore \int_{S_1} \vec{F} \cdot d\vec{a} = \int_{S_2} \vec{F} \cdot d\vec{a}$$

(c) Know that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \ \forall \ \vec{A}$.
Since we're given $\vec{\nabla}\cdot\vec{F} = 0$, it must be possible to write $\vec{F} = \vec{\nabla}\times\vec{A}$.

In E&M, we will find that $\vec{\nabla}\cdot\vec{B} = 0$.
$\therefore$ it is possible to write $\vec{B} = \vec{\nabla}\times\vec{A}$.
$\vec{A}$ will be called a vector potential.

3. $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\therefore \hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\therefore \hat{r}\cdot\vec{\nabla} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$\therefore (\hat{r}\cdot\vec{\nabla})\hat{r} =$$

$$\underbrace{\frac{1}{\sqrt{x^2+y^2+z^2}}}_{\frac{1}{r}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left[\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}\right]$$

Try just the x-derivative...

$$\left[\left(\hat{r}\cdot\vec{\nabla}\right)\hat{r}\right]_x = \frac{x}{r}\left[\frac{\hat{x}}{r} - \frac{\vec{r}\,\frac{1}{2}2x}{r^3}\right]$$

Get a similar result for y & z derivatives.

$$\therefore \left(\hat{r}\cdot\vec{\nabla}\right)\hat{r} = \frac{x}{r}\left[\frac{\hat{x}}{r} - \frac{\vec{r}\,x}{r^3}\right] + \frac{y}{r}\left[\frac{\hat{y}}{r} - \frac{\vec{r}\,y}{r^3}\right]$$

$$+ \frac{z}{r}\left[\frac{\hat{z}}{r} - \frac{\vec{r}\,z}{r^3}\right]$$

$$= \underbrace{\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r^2}}_{\frac{\vec{r}}{r^2} = \frac{\hat{r}}{r}} - \underbrace{\frac{\vec{r}}{r^4}\left(x^2 + y^2 + z^2\right)}_{\frac{\vec{r}}{r^4}r^2 = \frac{\vec{r}}{r^2} = \frac{\hat{r}}{r}}$$

$$\boxed{\therefore \left(\hat{r}\cdot\vec{\nabla}\right)\hat{r} = \frac{\hat{r}}{r} - \frac{\hat{r}}{r} = 0}$$

4. Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If $T = e^{-5x}\sin(4y)\cos(3z)$,

then $\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) e^{-5x}\sin(4y)\cos(3z)$

$$\frac{\partial T}{\partial x} = -5e^{-5x}\sin(4y)\cos(3z)$$

$$\frac{\partial^2 T}{\partial x^2} = 25e^{-5x}\sin(4y)\cos(3z) = 25T$$

$$\frac{\partial T}{\partial y} = 4e^{-5x}\cos(4y)\cos(3z)$$

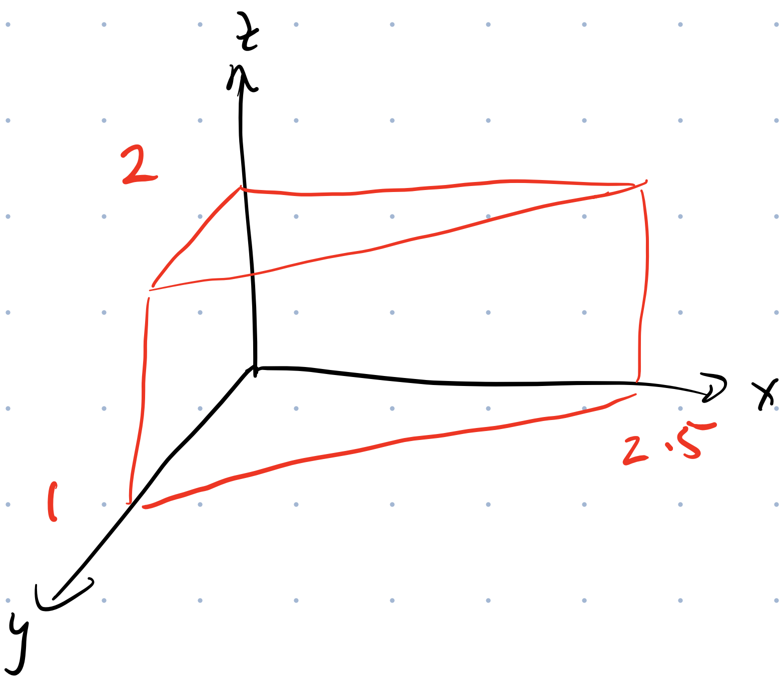
$$\frac{\partial^2 T}{\partial y^2} = -16e^{-5x}\sin(4y)\cos(3z) = -16T$$

$$\frac{\partial T}{\partial z} = -3e^{-5x}\sin(4y)\sin(3z)$$

$$\frac{\partial^2 T}{\partial z^2} = -9e^{-5x}\sin(4y)\cos(3z) = -9T$$

$$\boxed{\therefore \nabla^2 T = 25T - 16T - 9T = 0}$$

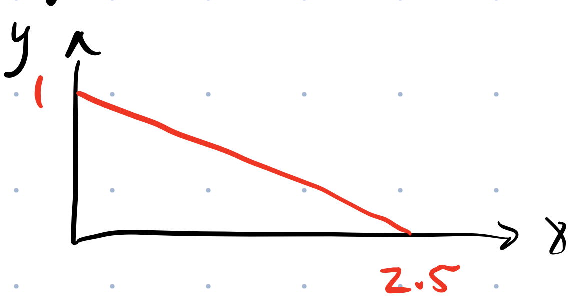
5. Calculate $\int T \, dx\,dy\,dz$ for $T = x^2 yz$ over the following volume.



(not drawn to scale).

$\therefore \quad 0 < z < 2$

Consider the xy-plane.



Equation of the line is

$$y = \underset{\text{y intercept}}{1} - \underset{\text{slope} = \frac{\text{rise}}{\text{run}}}{\frac{1}{2.5}} x = 1 - \frac{2x}{5}$$

$$\therefore \quad 0 < x < 2.5$$

$$0 < y < 1 - \frac{2x}{5}$$

$$\therefore I = \int I \, dx\,dy\,dz = \int_{z=0}^{2} \int_{x=0}^{2.5} \int_{y=0}^{1-\frac{2x}{5}} x^2 y z \, dy\,dx\,dz$$

$$= \int_{z=0}^{2} \int_{x=0}^{2.5} x^2 z \left(\int_{y=0}^{1-2x/5} y \, dy \right) dx\,dz$$

$$= \int_{z=0}^{2} \int_{x=0}^{2.5} x^2 z \; \frac{y^2}{2} \Bigg|_{y=0}^{1-2x/5} dx\,dz$$

$$= \frac{1}{2}\int_{z=0}^{2} z \int_{x=0}^{2.5} x^2 \left(1-\frac{2x}{5}\right)^2 dx\, dz$$

$$= \underbrace{\frac{1}{2}\int_{z=0}^{2} z\, dz}_{\frac{z^2}{2}\Big|_0^2 = 2} \int_{x=0}^{2.5} x^2\left(1-\frac{4x}{5}+\frac{4x^2}{25}\right) dx$$

$$= \int_{x=0}^{2.5}\left(x^2 - \frac{4x^3}{5} + \frac{4x^4}{25}\right) dx$$

$$= \left(\frac{1}{3}x^3 - \frac{x^4}{5} + \frac{4x^5}{125}\right)\Big|_0^{2.5}$$

$$= (2.5)^3\left(\frac{1}{3} - \underbrace{\frac{2.5}{5}}_{\frac{1}{2}} + \underbrace{\frac{4}{125}(2.5)^2}_{\frac{1}{5}}\right)$$

$$= (2.5)^3 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right)$$

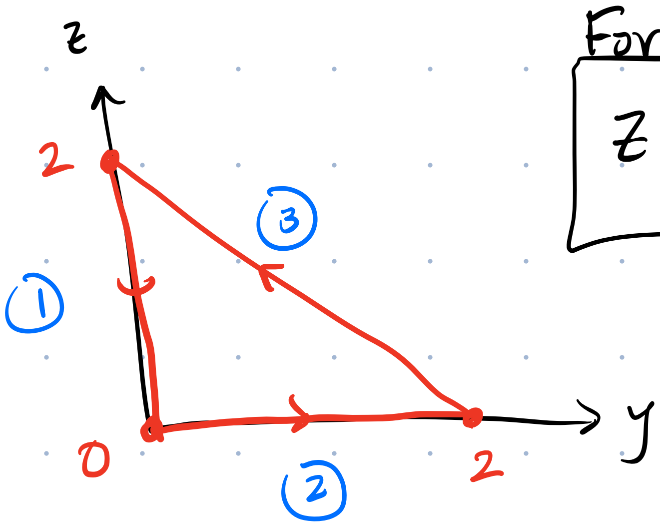
$$= (2.5)^3 \left(\frac{10 - 15 + 6}{30} \right) = \frac{(2.5)^3}{30}$$

$$\boxed{I = 0.521}$$

6. Stoke's Theorem:

$$\int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a} = \oint_P \vec{V} \cdot d\vec{l}$$

$$\vec{V} = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$$



For 3

$$\boxed{z = 2 - y}$$

$\hat{x}$ is out of screen/page.

Start w/ the surface integral.

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{x}\left(0 - 2y\right) - \hat{y}\left(3z - 0\right) + \hat{z}\left(0 - x\right)$$

$$= -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}$$

$$d\vec{a} = dy\,dz\,\hat{x}$$

$$\therefore \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{a} = -2y\,dy\,dz$$

$$\therefore I = \int\left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{a} = -2\int_{y=0}^{2} y \underbrace{\int_{z=0}^{2-y} dz}_{2-y}\,dy$$

$$I = -2\int_{y=0}^{2} y(2-y)\,dy = 2\int_{y=0}^{2} (y^2-2y)\,dy$$

$$= 2\left[\frac{y^3}{3} - y^2\right]\Bigg|_0^2$$

$$= 2\left(\frac{8}{3}-4\right) = 2\left(\frac{8-12}{3}\right) = \boxed{-\frac{8}{3}}$$

Next, we try the line integral:

$$I = \oint \vec{v}\cdot d\vec{l}$$

$$= \int_{①} \vec{v}\cdot d\vec{l} + \int_{②} \vec{v}\cdot d\vec{l} + \int_{③} \vec{v}\cdot d\vec{l}$$

① $d\vec{l} = -dz\,\hat{z}$

$\vec{v}\cdot d\vec{l} = x\,dz$, but $x=0$

$$\therefore \int_{①} \vec{V}\cdot d\vec{l} = \int_{z=0}^{2} 0\, dz = 0$$

② $d\vec{l} = dy\,\hat{y}$ $\quad \therefore \vec{V}\cdot d\vec{l} = -3z\,dy$

but $z=0$

$$\therefore \int_{②} \vec{V}\cdot d\vec{l} = \int_{x=0}^{2} 0\, dy = 0$$

③ For part ③ of the path, recall that

$z = 2-y$. & $x=0$

$\therefore \vec{V} = 2y(2-y)\,\hat{y}$

since $dz = -dy$, we can integral over the interval $0<y<2$ w/ $d\vec{l} = -dy\,\hat{y}$

$$\therefore \vec{v} \cdot d\vec{l} = -2(2y - y^2)\, dy$$

$$\therefore I = -2 \int_{y=0}^{2} (2y - y^2)\, dy$$

$$= -2 \left[y^2 - \frac{y^3}{3} \right] \Big|_0^2$$

$$= -2 \left[4 - \frac{8}{3} \right] = -2 \left[\frac{12-8}{3} \right]$$

$$= -2 \left[\frac{4}{3} \right] = \boxed{-\frac{8}{3}}$$

same as before!